A New Twist on the Marcum Q-Function and its Application

Marvin K. Simon

Abstract

A new form of the Marcum Q-function is presented that has both computational and analytical advantages. The new form is particularly useful in simplifying and rendering more accurate the analysis of the error probability performance of uncoded and coded partially coherent, differentially coherent, and noncoherent communication systems in the presence of fading. It also enables simple upper and lower bounds to be found analogous to the Chernoff bound on the Gaussian Q-function.

[1]This work was performed at the Jet Propulsion Laboratory, California Institute of Technology under a contract with the National Aeronautics and Space Administration. The author is with the Jet Propulsion Laboratory.

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1. Introduction

In a recent paper [1]. a particular form of the Gaussian probability integral $Q(x)$ developed a number of years ago by Craig [21 was used to simplify and render more accurate a number of performance results related to communication problems dealing with coherent detection, in particular, those where the argument of Q(x) is dependent on random system parameters and thus requires averaging over the statistics of these parameters. More specifically, Craig showed that the Gaussian probability function could be expressed as the definite integral

$$Q(x)=\frac{1}{\pi}\int_0^{\pi/2}\exp\left(-\frac{x^2}{2\sin^2\theta}\right)d\theta, \qquad x\geq 0 \tag{1}$$

which, in addition to the advantage of having finite integration limits, had its argument contained in the *integrand* rather than in the *integration limits* as in the traditional definition of the function. By noting that the integrand has its maximum value when $\theta=\pi/2$, then, replacing the integrand by its maximum value one readily obtains the well-known upper bound on Q(x), namely,

$$Q(x)\leq\tfrac{1}{2}\exp\left(-x^2/2\right) \tag{2}$$

which ‘is in the form of a Chernoff bound. Comparing (1) with (2), it was further shown in [1] that the form of Eq. (1) allowed manipulations akin to those afforded by the Chernoff bound in (2) but without the necessity of invoking such a bound.

Motivated by Craig’s work, we set out to see if a similar form could be obtained for the Marcum Q-function [3] which is common in performance results for communication problems related to partially coherent, differentially coherent, and noncoherent communications [4-7]. This paper develops such a desirable form and then discusses how it might be applied.

2. A New Form of the Generalized Marcum O-Function

The generalized Marcum Q-function is defined by the integral

$$Q_M(\alpha,\beta) = \frac{1}{\alpha^{M-1}}\int_\beta^\infty x^M \exp\left[-\left(\frac{x^2+\alpha^2}{2}\right)\right] I_{M-1}(\alpha x)dx \tag{3}$$

which also has the series form [5,6]

$$Q_M(\alpha,\beta) = \exp\left(-\frac{\alpha^2+\beta^2}{2}\right)\sum_{k=1-M}^{\infty}\left(\frac{\alpha}{\beta}\right)^k I_k(\alpha\beta) = \exp\left(-\frac{\beta^2}{2}\left[1+\zeta^2\right]\right)\sum_{k=1-M}^{\infty}\zeta^k I_k(\beta^2\zeta), \quad \beta>\alpha\geq 0 \tag{4}$$

where $\zeta \triangleq \alpha/\beta$. The reason for introducing the parameter ζ to represent the ratio of the variables of the Marcum Q-function will be explained later on when dealing with the applications. The modified Bessel function of kth order can be expressed as the integral [8]

$$I_k(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(-je^{-j\theta}\right)^k e^{-z\sin\theta}d\theta \tag{5}$$

where $j=\sqrt{-1}$ and it is clear that the imaginary part of the right hand side of (5) must be equal to zero (since $I_k(z)$ is a real function of the real argument z.) Using (5) in (4) and noting from the inequality on the arguments of the Marcum Q-function given in (1) that 0< $\zeta<1$, we obtain

$$\begin{aligned} Q_M(\alpha,\beta) &= \exp\left(-\frac{\beta^2}{2}\left[1+\zeta^2\right]\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1-M}^{\infty}\zeta^k\left(-je^{-j\theta}\right)^{|k|}e^{-\beta^2\zeta\sin\theta}d\theta \\ &= \exp\left(-\frac{\beta^2}{2}\left[1+\zeta^2\right]\right)\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{1+je^{-j\theta}\zeta}+\frac{\left(-je^{-j\theta}\zeta^{-1}\right)\left[1-\left(-je^{-j\theta}\zeta^{-1}\right)^{M-1}\right]}{1+je^{-j\theta}\zeta^{-1}}\right]e^{-\beta^2\zeta\sin\theta}d\theta \end{aligned} \tag{6}$$

where we have also used the fact that $I_k(\beta^2\zeta) = I_{-k}(\beta^2\zeta)$. Rationalizing the denominators of the complex factors between brackets of the integrand and recognizing again that the imaginary part of the results must result in a zero integral (since $Q(\alpha,\beta)$ *is* real), gives after much simplification the desired form

$$Q_M(\alpha,\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{(\ )^{\frac{M-1}{2}}\zeta^{-(M-1)}(\cos(M-1)6+\gamma\sin M\theta)}{1+2\zeta\sin\theta+\zeta^2}\right]\exp\left(-\frac{\beta^2}{2}\left[1+2\zeta\sin\theta+\zeta^2\right]\right)d\theta,$$

$$\zeta=\alpha/\beta<1, \quad M \text{ odd}$$

$$Q_M(\alpha,\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{(-1)^{\frac{M}{2}}\zeta^{-(M-1)}(\sin(M-1)\theta-\zeta\cos M\theta)}{1+2\zeta\sin\theta+\zeta^2}\right]\exp\left(-\frac{\beta^2}{2}\left[1+2\zeta\sin\theta+\zeta^2\right]\right)d\theta,$$

$$\zeta=\alpha/\beta<1, \quad M \text{even}$$

(7)

which has the popular special case $(M = 1)$

$$Q_1(\alpha,\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1+\zeta\sin\theta}{1+2\zeta\sin\theta+\zeta^2}\right]\exp\left(-\frac{\beta^2}{2}\left[1+2\zeta\sin\theta+\zeta^2\right]\right)d\theta, \qquad \beta > \alpha \geq 0$$

(8)

For the case a $\geq \beta \geq$ O, the appropriate series form is [5,6]

$$Q_M(\alpha,\beta) = 1 - \exp\left(-\frac{\alpha^2+\beta^2}{2}\right)\sum_{k=M}^{\infty}\left(\frac{\beta}{\alpha}\right)^k I_k(\alpha\beta) = 1 - \exp\left(-\frac{\alpha^2}{2}\left[1+\zeta^2\right]\right)\sum_{k=M}^{\infty}\zeta^k I_k(\alpha^2\zeta)$$

(9)

whereupon an analogous development would yield the result

$$Q_M(\alpha,\beta) = 1 + \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{(-1)^{\frac{M-1}{2}}\zeta^M(\sin M\theta + \zeta\cos(M-1)\theta)}{1+2\zeta\sin\theta+\zeta^2}\right]\exp\left(-\frac{\alpha^2}{2}\left[1+2\zeta\sin\theta+\zeta^2\right]\right)d\theta,$$

$$\zeta = \beta/\alpha < 1, \quad M \text{ odd}$$

$$Q_M(\alpha,\beta) = 1 - \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{(-1)^{\frac{M}{2}}\zeta^M(\cos M\theta - \zeta\sin(M-1)\theta)}{1+2\zeta\sin\theta+\zeta^2}\right]\exp\left(-\frac{\alpha^2}{2}\left[1+2\zeta\sin\theta+\zeta^2\right]\right)d\theta,$$

$$\zeta = \beta/\alpha < 1, \quad M \text{even}$$

(lo)

and where now $\zeta \triangleq \beta / a < 1$. Once again the special case of M = 1 becomes[1]

$$Q_1(\alpha,\beta) = 1 + \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{\zeta^2+\zeta\sin\theta}{1+2\zeta\sin\theta+\zeta^2}\right]\exp\left(-\frac{\alpha^2}{2}\left[1+2\zeta\sin\theta+\zeta^2\right]\right)d\theta, \qquad \alpha > \beta \geq 0 \qquad (11)$$

We note that similar to (l), the expression in (7) (or (10)) is a single integral with finite limits and an integrand that is bounded and well-behaved over the interval $-\pi \leq \theta \leq \pi$ and is exponential (Gaussian) in one of the arguments. Aside from its analytical desirability in the applications to be discussed next, the form of (7) (or (10)) is also computationally desirable relative to other methods [9,10] previously reported *in* the literature for numerically evaluating the Marcum Q- function.

Simple upper and lower bounds on $Q_1(\alpha, \beta)$ can be obtained in the same manner that (2) was obtained from (1). In particular, for $\beta > a \geq 0$, ***we*** observe that

[1]It might appear from (11) that the Marcum-Q function can exceed unity. However, the integral in (11) is always less than or equal to zero. Furthermore, the special case of $a = \beta$ has the closed form result $Q_1(\alpha,\alpha) = \left[1+\exp(-\alpha^2)I_0(\alpha^2)\right]/2$ [16, Eq. (A-3-2)].

the maximum and minimum of the integrand in (8) occurs for $\theta = -\pi/2$ and $\theta = \pi/2$ respectively. Thus, replacing the integrand by its maximum and minimum values leads to the upper and lower "Chernoff-type" bounds

$$\frac{\beta}{\beta+\alpha}\exp\left(-\frac{(\beta+\alpha)^2}{2}\right) \le Q_1(\alpha,\beta) \le \frac{\beta}{\beta-\alpha}\exp\left(-\frac{(\beta-\alpha)^2}{2}\right) \tag{12}$$

which is asymptotically tight as $\alpha \to 0$.

For $\alpha > \beta \ge 0$, the integrand in (11) has a minimum at $\theta = -\pi/2$ and a maximum at $\theta = \pi/2$. Since the maximum of the integrand, namely, $(\zeta/(1+\zeta))\exp(-\alpha^2(1+\zeta)^2/2)$, is always positive, then the upper bound obtained by replacing the integrand by this value would exceed unity and hence be useless. On the other hand, the minimum of the integrand, namely, $-(\zeta/(1-\zeta))\exp(-\alpha^2(1-\zeta)^2/2)$ is always negative, Hence, a lower "Chernoff-type" bound on $Q_1(\alpha,\beta)$ is given by

$$1-\frac{\alpha}{\alpha-\beta}\exp\left(-\frac{(\alpha-\beta)^2}{2}\right) \le Q_1(\alpha,\beta) \tag{13}$$

We now proceed to briefly discuss the application of this new form of the Marcum Q- function to communication problems where α and β are both proportional to the square root of signal-to-noise ratio (SNR) but their ratio, i.e., ζ, is independent of SNR.

3. Error Probability Performance of Binary Communications in the Presence of Fading Amplitudes

Many problems dealing with the error probability performance of partially coherent, differentially coherent, and noncoherent detection of PSK and FSK signals in additive white Gaussian noise (AWGN) have an error probability expression of the form [4-7]

$$P(E) = \frac{1}{2}\left[1 - Q\left(\sqrt{b},\sqrt{a}\right) + Q\left(\sqrt{a},\sqrt{b}\right)\right] = Q\left(\sqrt{a},\sqrt{b}\right) - \frac{1}{2}\exp\left[-\frac{1}{2}(a+b)\right] I_0\left(\sqrt{ab}\right) \tag{14}$$

where a, b are each proportional to SNR with $b > a$. In view of (8) and (11), the error probability in (14) can be expressed as the single integral

$$P(E) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\left[\frac{1-\zeta^2}{1+2\zeta\sin\theta+\zeta^2}\right]\exp\left(-\frac{b}{2}\left[1+2\zeta\sin\theta+\zeta^2\right]\right)d\theta, \qquad \zeta = \sqrt{\frac{a}{b}} < 1 \tag{15}$$

Again note that the maximum and minimum of the integrand in (15) occurs for $\theta = -\pi / 2$ and $\theta = \pi / 2$ respectively. Thus, replacing the integrand by its maximum and minimum values leads to the simple upper and lower bounds

$$\frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{a}}\exp\left(-\frac{\left(\sqrt{b}+\sqrt{a}\right)^2}{2}\right) \le P(E) \le \frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}\exp\left(-\frac{\left(\sqrt{b}-\sqrt{a}\right)^2}{2}\right) \tag{16}$$

To evaluate the average error probability performance of such systems in the presence of slow fading, one would multiply the arguments $\sqrt{a},\sqrt{b}$ in the Marcum Q-functions of (14) by the normalized (unit power) fading parameter ρ and then average the resulting conditional (on ρ) error probability expression over the probability density function (pdf) of ***p***, namely, p_ρ ***(p).*** Using the form of the error probability in (15) with b replaced by ***p2b,*** then since the integral over ***p*** can in most cases, e.g., Rayleigh, Rice, Nakagami fading, be evaluated in closed form, the resulting error probability is in the form of a single integral with finite limits and an integrand composed of elementary functions.

Without going into great detail (because of space limitation) suffice it to say that the new representation of (7) and (10) allows for a unified approach to partially coherent, differentially coherent, and noncoherent modulations communicated over generalized fading single- and multi-channels. A complete and exhaustive treatment of this entire subject will be presented in a forthcoming paper [13] by the author and one of his colleagues. Furthermore, using the results introduced in [11, the unified approach can also be made to include coherent systems [14]. As such, a tutuorial/survey paper [15] has been written that will present a unified approach to the performance analysis of digital communication over generalized fading channels.

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